JOURNAL OF APPROXIMATION THEORY 19, 154–175 (1977)

General Function Spaces. II. Inequalities of Plancherel–Polya–Nikol'skij-Type, L_p-Spaces of Analytic Functions, $0 < p \leqslant \infty$

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Communicated by P. L. Butzer

Received October 6, 1975

1. INTRODUCTION

This is the second part of a series of papers on general function spaces. The first part [11] contains a description of abstract decomposition methods. This part deals with the basic spaces from which the more general spaces, considered in later papers, are built up: weighted L_p-spaces of analytic functions, $0 < p \leqslant \infty$. This part is self-contained.

The aim of this paper is the proof of inequalities of Plancherel–Polya–Nikol'skij type and the consideration of related quasi-Banach spaces. Let $\rho(x)$ be a weight function, and let μ_1 and μ_2 be two Borel measures in R_n with some additional properties explained later on. Then we shall be concerned with inequalities of type

$$\text{(I)} \qquad \| \rho f \|_{L_q,\mu_1} \leqslant c \| \rho f \|_{L_p,\mu_2}, \qquad \infty \geqslant q \geqslant p > 0,$$

and of type

$$\text{(II)} \qquad \| \rho D^\alpha f \|_{L_q,\mu_1} \leqslant c \| \rho f \|_{L_p,\mu_2}, \qquad \infty \geqslant q \geqslant p > 0.$$

Here f belongs to a set of entire analytic functions in R_n where the supports of the corresponding Fourier transforms are contained in a fixed compact set in R_n. $L_{p,\mu}$ means the usual quasi-Banach space in R_n with respect to the measure μ. For $\rho(x) \equiv 1$, $d\mu_1 = d\mu_2 = dx =$ Lebesgue measure and $1 \leqslant p \leqslant q \leqslant \infty$, one obtains the classical inequalities of Plancherel–Polya–Nikol'skij type, which play a fundamental role in the theory of function spaces, namely in the approach given by Nikol'skij [6] (approximation of functions by entire analytic functions of exponential type). The corresponding L_p-spaces treated in this paper are

$$\{f \mid f \in (S_a)', \operatorname{supp} Ff \subset \bar{\Omega}, \| \kappa f \|_{L_{p,\mu}} < \infty\},$$

ISSN 0021-9045

$0 < p \leqslant \infty$. Here $(S_a)'$ is a space of distributions (lying between S' and D'), Ff is the Fourier transform of f, and Ω is a fixed bounded domain, κ is a weight function, and μ a measure.

We extend the classical inequalities in two directions: first to $0 < p < 1$, and second to weight functions $\rho(x)$ and measures μ. For the first extension we need inequalities of maximal type (Hardy's maximal function), for the second one ultra-distributions. In the Appendix (Section 5) some facts on ultra-distributions needed are described; in particular a proof of a theorem of Paley–Wiener–Schwartz type (Theorem 5.5) is given which is perhaps of interest in itself. Apart from this theorem the main results of the paper are contained in Theorem 3.5 (resp. 3.6) and Theorem 4.1.

All unimportant positive numbers in this paper are denoted by the same letters $c, c', \ldots$.

2. Definitions

In this section we give the definitions for the weight function $\rho(x)$, the measures, and the L_p-spaces mentioned in the Introduction. We shall use the notations introduced in the Appendix (Section 5) of this paper.

2.1. *Weight Functions*

R_n denotes the n-dimensional real Euclidean space.

Definition 2.1. Let $1 < a < \infty$, and let $\{C_\epsilon\}_{\epsilon > 0}$ be a set of positive numbers. Then $K(a, C_\epsilon)$ denotes the set of all Borel-measurable functions $\rho(x)$ in R_n such that

$$0 < \rho(x) \leqslant C_\epsilon \rho(y) \exp \epsilon \mid x - y \mid^{1/a} < \infty \tag{1}$$

for all $\epsilon > 0$, $x \in R_n$, and $y \in R_n$.

Remarks. (1) Setting $y = 0$ in (1), it follows by Lemma 5.1 that

$$\rho(x) \in (S_a)'. \tag{2}$$

Equation (1) may be simplified. Let

$$0 < \rho(x) \leqslant c\rho(y) \exp c' \mid x - y \mid^{\beta}, \qquad c > 0,\ c' > 0,\ 0 < \beta < 1. \tag{3}$$

If the Borel-measurable function $\rho(x)$ satisfies (3), then it also satisfies (1), provided $\beta < 1/a < 1$. On the other hand, choosing $\epsilon = 1$ in (1), one obtains (3) with $\beta = 1/a$. But the advantage of the more complicated condition (1) is the relation (2) which will be useful later on.

(2) For all $\epsilon > 0$ it follows immediately from (1) that

$$c_\epsilon \exp(-\epsilon \mid x \mid^{1/a}) \leqslant \rho(x) \leqslant c_\epsilon' \exp \epsilon \mid x \mid^{1/a}, \qquad c_\epsilon > 0, \ c_\epsilon' > 0. \tag{4}$$

Consequently, the growth of $\rho(x)$ is restricted from above and from below. Furthermore, as an easy consequence of (1) one obtains the following assertion: Let $\lambda > 0$, $\rho_1 \in K(a, C_\epsilon)$, $\rho_2 \in K(a, C_\epsilon')$. Then

$$\begin{aligned} &\lambda\rho_1 \in K(a, C_\epsilon), \qquad \rho_1{}^\lambda \in K(a, C_{\epsilon/\lambda}^\lambda), \\ &\rho_1 + \rho_2 \in K(a, \max(C_\epsilon, C_\epsilon')), \\ &1/\rho_1 \in K(a, C_\epsilon), \\ &\rho_1\rho_2 \in K(a, C_{\epsilon/2}C'_{\epsilon/2}), \qquad \rho_1/\rho_2 \in K(a, C_{\epsilon/2}C'_{\epsilon/2}). \end{aligned} \tag{5}$$

EXAMPLE. If $\alpha \geqslant 0$ and $j = 1,..., n$, then

$$1 + \mid x_j \mid^\alpha \in K(a, C_\epsilon) \tag{6}$$

for all $1 < a < \infty$ and appropriate C_ϵ, depending on a. If $0 \leqslant \beta < 1$, then

$$e^{|x_j|^\beta} \in K(a, C_\epsilon) \tag{7}$$

for $\beta < 1/a < 1$ and appropriate C_ϵ. Using (5) one can construct a large variety of functions ρ of type (1), for instance frational functions of (6) and (7) with positive coefficients.

2.2. *Measures*

Let $h > 0$, and let

$$Q_k{}^h = \{x \mid x = (x_1, ..., x_n), hk_j \leqslant x_j < h(k_j + 1), (j = 1,..., n)\} \tag{8}$$

be a decomposition of R_n where $k = (k_1, ..., k_n)$, k_j being integers.

DEFINITION 2.2. M_h denotes the set of all Borel measures in R_n such that

$$\mu(Q_k{}^h) = 1 \qquad \text{for all } Q_k{}^h. \tag{9}$$

Remark. The last definition shows that we shall be concerned with measures having a lattice structure. The following two measures are of special interest: (i) the modified Lebesgue measure dx/h^n, and (ii) atomic measures $\mu(Q_k{}^h) = \mu(\{x^k\}) = 1$, where x^k is a fixed point in $Q_k{}^h$. The inequalities proved later on are independent of $\mu \in M_h$, provided that h is sufficient small.

2.3. *Admissible Weight Functions and L_p-Spaces*

If μ is a Borel measure in R_n, then $\|\cdot\|_{L_{p,\mu}}$ has the usual meaning, namely

$$\begin{aligned} \|f\|_{L_{p,\mu}} &= \left(\int_{R_n} |f(x)|^p \, d\mu\right)^{1/p} \qquad \text{for} \quad 0 < p < \infty, \\ &= \mu - \text{ess sup} \, |f(x)| \qquad \text{for} \quad p = \infty, \end{aligned} \tag{10}$$

$f(x)$ are complex-valued functoins. F denotes the Fourier transform (Appendix 5.4).

DEFINITION 2.3. Let $\rho \in K(a, C_\epsilon)$, and let $\mu \in M_h$.

(a) A Borel-measurable function $\kappa(x)$ is said to be admissible (with respect to ρ and μ) if

(i) there exists a positive number c such that $0 \leqslant \kappa(x) \leqslant c\rho(x)$ for all $x \in R_n$,

(ii) there exist a positive number δ, a positive number c', and a Borel-measurable subset G of R_n such that $\mu(G \cap Q_k^h) \geqslant \delta$ for all Q_k^h and

$$\kappa(x) \geqslant c'\rho(x) \qquad \text{for} \quad x \in G, \tag{11}$$

(b) Let $\kappa(x)$ be an admissible function (with respect to ρ and μ), let $0 < p \leqslant \infty$, and let Ω be a bounded subset of R_n. Then

$$L_p^\Omega(\kappa, \mu) = \{f \mid f \in (S_a)', \operatorname{supp} Ff \subset \bar{\Omega}, \|\kappa f\|_{L_{p,\mu}} < \infty\}. \tag{12}$$

Remark. Later it will be shown that all the spaces $L_p^\Omega(\kappa, \mu)$ are quasi-Banach spaces, provided that h is sufficiently small.

EXAMPLE. The most interesting feature of the last definition is the possibility to replace $\rho \in K(a, C_\epsilon)$ by κ. Let $d\mu = dx/h^n$ be the modified Lebesgue measure. Using the examples in Section 2.1 it follows that

$$\kappa(x) = \prod_{j=1}^{n} |x_j|^{\beta_j}, \qquad \beta_j \geqslant 0$$

and

$$\kappa(x) = |x|^\beta, \qquad \beta \geqslant 0$$

are admissible functions (where $\rho(x) = \prod_{j=1}^n (1 + |x_j|)^{\beta_j}$ and $\rho(x) = 1 + |x|^\beta$, respectively). Of particular interest seems to be the case $\kappa(x) = |x_n|^\beta$; $\beta \geqslant 0$.

3. Inequalities

In this section inequalities of types (I) and (II) mentioned in the Introduction are proved. It will be convenient to divide the proof of the general inequalities into two main steps: First, the inequalities are proved for Lebesgue measures. Here the lattice structure of the measures does not play any role (see Section 3.1, 3.3). Second, an equivalence theorem will be derived showing that one may replace the Lebesgue measure by an arbitrary measure $\mu \in M_h$ (see Section 3.4). Afterward it is not hard to formulate the general results (see Sections 3.5 and 3.6). The question whether the assumptions made are natural ones is discussed in Section 3.7.

3.1. *Inequalities of Type* (I) *for the Lebesgue Measure*

To avoid technical difficulties we first prove the inequalities for rapidly decreasing analytic functions. We use the notations and the results of the appendix. Let $\|\cdot\|_{L_p} = \|\cdot\|_{L_{p,\mu}}$ if $d\mu = dx$ is the Lebesgue measure.

THEOREM 3.1. *Let $\rho \in K(a, C_\epsilon)$, $b > 0$, and $0 < p \leqslant q \leqslant \infty$. Then there exists a positive number C such that for all entire analytic functions f,*

$$f \in S_a\,, \qquad \operatorname{supp} Ff \subset \{y \mid |y| \leqslant b\}, \tag{13}$$

one has

$$\|\rho f\|_{L_q} \leqslant C\,\|\rho f\|_{L_p}\,. \tag{14}$$

Proof. *Step* 1. It follows from (64) and (4) that both sides of (14) are finite. Let $\psi \in S_a$ such that $F\psi$ has compact support and $(F\psi)(x) = 1$ for $|x| \leqslant b$. The existence of such a function follows from (56) and the remarks in Section 5.3. If f satisfies (13) it follows that $Ff = Ff \cdot F\psi$, and consequently

$$f(x) = c \int_{R_n} f(y)\,\psi(x-y)\,dy. \tag{15}$$

Therefore, by (1),

$$\rho(x)\,|f(x)| \leqslant c_\epsilon \int_{R_n} |f(y)|\,|\psi(x-y)|\,\rho(y)\exp(\epsilon\,|x-y|^{1/a})\,dy. \tag{16}$$

If $1 \leqslant p < \infty$, then again using (64) and choosing ϵ sufficiently small, it follows by Hölder's inequality that

$$\|\rho f\|_{L_\infty} \leqslant c\,\|\rho f\|_{L_p}\,. \tag{17}$$

If $0 < p < 1$, (15) yields

$$\rho(x)\,|f(x)| \leqslant c'(\sup \rho(y)\,|f(y)|)^{1-p} \int_{R_n} \rho^p(y)\,|f(y)|^p\,dy.$$

Taking the supremum on the left-hand side and using $\|\rho f\|_{L_\infty} < \infty$, one obtains (17) for $0 < p < 1$.

Step 2. Let $0 < p < q < \infty$. Then (14) is a consequence of (17) and

$$\|\rho f\|_{L_q} \leqslant \|\rho f\|_{L_\infty}^{1-p/q} \|\rho f\|_{L_p}^{p/q}.$$

Remark. The case $\rho(x) \equiv 1$ reproduces the classical Plancherel–Polya–Nikol'skij inequalities.

3.2. *Inequalities of Maximal Type*

For the proof of inequalities of type (II) mentioned in the Introduction, some preliminaries are needed. Our approach is based upon the technique of maximal inequalities developed by Fefferman and Stein [1] and Peetre [7]. In particular, some ideas of the proof of the lemma below are taken from Peetre [7]. As usual, $(Mf)(x)$ denotes Hardy's maximal function,

$$(Mf)(x) = \sup_B (1/|B|) \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B centered at $x \in R_n$.

LEMMA 3.2. *Let $\rho \in K(a, C_\epsilon)$, $b > 0$, and $0 < r < \infty$. Then there exists a positive number C such that for all functions of type* (13)

$$\sup_{z \in R_n} \rho(x-z) \frac{|\nabla f(x-z)|}{1+|z|^{n/r}} \leqslant C[(M|\rho f|^r)(x)]^{1/r}. \tag{18}$$

Proof. If ψ is as in the proof of Theorem 3.1, then $\partial\psi/\partial x_1$ belongs to also S_a and its Fourier transform has a compact support. Hence by (15), (64), and the counterpart to (16),

$$\rho(x-z) \left| \frac{\partial f}{\partial x_1}(x-z) \right| \leqslant c \int_{R_n} \rho(y)\, |f(y)| \exp(-\lambda\, |x-z-y|^{1/a})\, dy,$$

where λ is an appropriate positive number. Using the estimate

$$(1+|x-y|^{n/r})/(1+|z|^{n/r}) \leqslant c(1+|x-y-z|^{n/r}),$$

it follows that

$$\begin{aligned} \sup_z \rho(x-z) \frac{|(\partial f/\partial x_1)(x-z)|}{1+|z|^{n/r}} &\leqslant c' \sup_z \int_{R_n} \rho(y) \frac{|f(y)|}{1+|x-y|^{n/r}} \\ &\qquad \times \exp(-\lambda'\, |x-z-y|^{1/a})\, dy \\ &\leqslant c'' \sup_w \rho(x-w) \frac{|f(x-w)|}{1+|w|^{n/r}}, \end{aligned} \tag{19}$$

λ' being a positive number with $\lambda' < \lambda$. Now we use the fact that

$$\sup_{|v|\leqslant 1} |\nabla g(v)| + \left(\int_{|v|\leqslant 1} |g(v)|^r \, dv\right)^{1/r} \tag{20}$$

is an equivalent quasi-norm in the Banach space $C^1(\{v \mid |v| \leqslant 1\})$ (a proof of this assertion will be given in the remark below). Let B_δ be a ball of radius δ. By a homogeneity argument it follows for $g \in C^1(B_\delta)$ that for $v \in B_\delta$

$$|g(v)| \leqslant c\delta \sup_{w\in B_\delta} |\nabla g(w)| + c\delta^{-n/r} \left(\int_{B_\delta} |g(w)|^r \, dw\right)^{1/r}, \tag{21}$$

where c is independent of δ. Now we apply (21) to $f(x - w)$. If one assumes $\delta \leqslant 1$ and takes into consideration that

$$c_1\rho(u) \leqslant \rho(v) \leqslant c_2\rho(u), \qquad |u - v| \leqslant 1, \tag{22}$$

where c_1 and c_2 are two positive numbers independent of $u \in R_n$ and $v \in R_n$, then it follows that

$$\begin{aligned}\rho(x - w) |f(x - w)| \leqslant{}& c\delta \sup_{|y|\leqslant\delta} \rho(x - w - y) |\nabla f(x - w - y)| \\ &+ c\delta^{-n/r} \left(\int_{|y|\leqslant\delta} \rho^r(x - w - y) |f(x - w - y)|^r \, dy\right)^{1/r}.\end{aligned} \tag{23}$$

The integral is estimated from above by

$$\left(\int_{|u|\leqslant|w|+1} \rho^r(x - u) |f(x - u)|^r \, du\right)^{1/r} \leqslant c(1 + |w|^{n/r})[M |\rho f|^r)(x)]^{1/r}.$$

Putting this estimate in (23), dividing both sides by $1 + |w|^{n/r}$ and taking the supremum with respect to $w \in R_n$, it follows that

$$\begin{aligned}\sup_w \rho(x - w) \frac{|f(x - w)|}{1 + |w|^{n/r}} \leqslant{}& c\delta \sup_w \rho(x - w) \frac{|\nabla f(x - w)|}{1 + |w|^{n/r}} \\ &+ c\delta^{-n/r}(M |\rho f|^r(x))^{1/r},\end{aligned} \tag{24}$$

where c is independent of δ. Obviously, one may replace $\partial f/\partial x_1$ in (19) by ∇f. Choosing δ in (24) sufficiently small, then (18) is a consequence of (24) and (19), where $\partial f/\partial x_1$ is replaced by ∇f.

Remark. We used the fact that (20) is an equivalent quasi-norm in the Banach space C^1 of all continuously differentiable functions in the closed unit ball B. We give a proof. Obviously, (20) can be estimated from above by $\| g \|_{C^1(B)}$. The converse inequality is a consequence of

$$\begin{aligned}|g(v)| &\leqslant c(\min_{|w|\leqslant 1} |g(w)| + \sup_{|w|\leqslant 1} |\nabla g(w)|) \\ &\leqslant c\left[\left(\int_{|w|\leqslant 1} |g(w)|^r \, dw\right)^{1/r} + \sup_{|w|\leqslant 1} |\nabla g(w)|\right]\end{aligned} \tag{25}$$

which follows from the mean value theorem.

3.3. *Inequalities of Type* (II) *for the Lebesgue Measure*

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, α_j being nonnegative integers, then $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ has the usual meaning with $|\alpha| = \sum_{j=1}^n \alpha_j$.

THEOREM 3.3. *Let $\rho \in K(a, C_\epsilon)$, $b > 0$, $0 < p \leqslant q \leqslant \infty$, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index. Then there exists a positive number C such that for all functions f as given via* (13) *there holds*

$$\| \rho D^\alpha f \|_{L_q} \leqslant C \| \rho f \|_{L_p}. \tag{26}$$

Proof. The case $p = q = \infty$ follows after differentiation of (15) with respect to x in the same way as in the proof to Theorem 3.1. Let $0 < p = q < \infty$. Choosing $0 < r < p$, then (18) yields

$$\| \rho(\partial f/\partial x_1)\|_{L_p} \leqslant c \|(M | \rho f |^r)^{1/r} \|_{L_p} = c \| M | \rho f |^r \|_{L_{p/r}}^{1/r}.$$

By Hardy's maximal inequality (see [10, p. 5]) it follows that

$$\| \rho(\partial f/\partial x_1)\|_{L_p} \leqslant c \| | \rho f |^r \|_{L_{p/r}}^{1/r} = c \| \rho f \|_{L_p}. \tag{27}$$

If f satisfies (13), then $\partial f/\partial x_1$ does, too. Consequently, by iteration of the last estimate, where x_1 can be replaced by x_j, (26) follows with $0 < p = q < \infty$. Since $D^\alpha f$ satisfies (13), too, the general case $0 < p \leqslant q \leqslant \infty$ is a consequence of Theorem 3.1.

Remark. If $\rho(x) \equiv 1$ and $1 \leqslant p \leqslant q \leqslant \infty$, then (26) is used by Nikol'skij [6] in connection with the theory of function spaces.

3.4. *An Equivalence Theorem*

THEOREM 3.4. *Let $\rho \in K(a, C_\epsilon)$, $b > 0$, $0 < p \leqslant \infty$, and $\mu_1, \mu_2 \in M_h$. If $\kappa_1(x)$ is an admissible function with respect to ρ and μ_1, and if $\kappa_2(x)$ is an admissible function with respect to ρ and μ_2 (see Definition* 2.3(a)), *then there exists a positive number C such that for all functions f of type* (13)

$$\| \kappa_1 f \|_{L_{p,\mu_1}} \leqslant C \| \kappa_2 f \|_{L_{p,\mu_2}}, \tag{28}$$

provided that h is sufficiently small (*that means $0 < h \leqslant h_0$, where h_0 depends only on n, p, a, b, and C_ϵ, but not on μ_1, μ_2, $\kappa_1(x)$ and $\kappa_2(x)$*).

Proof. Step 1. First we prove inequalities of type (28) for the modified Lebesgue measure and the atomic measures described in the Remark in Section 2.2. More precisely, there exist two positive numbers c_1 and c_2 such that for all functions f satisfying (13), there holds

$$c_1 \| \rho(x^k) f(x^k)\|_{l_p} \leqslant \| \rho f \|_{L_p} h^{-n/p} \leqslant c_2 \| \rho(x^k) f(x^k)\|_{l_p}. \tag{29}$$

Here $x^k \in Q_k{}^h$ are arbitrary points, and

$$\| a_k \|_{l_p} = \Big(\sum_{k=(k_1,\ldots,k_n)} | a_k |^p \Big)^{1/p}, \qquad 0 < p < \infty,$$

with the usual modification for $p = \infty$. The constants c_1 and c_2 are independent of h and of the choice of $x^k \in Q_k{}^h$. If $x \in Q_k{}^h$, then

$$| f(x)| \leqslant | f(x^k)| + ch \sup_{z \in Q_k{}^h} | \nabla f(z)|. \tag{30}$$

Restricting h by $h \leqslant 1$ and using an inequality of type (22), it follows that

$$\rho(x) \, | f(x)| \leqslant c\rho(x^k) \, | f(x^k)| + ch \sup_{|x-z| \leqslant c'} \rho(z) \, | \nabla f(z)|.$$

Here c and c' are independent of h. If $p < \infty$, then

$$\int_{R_n} \rho^p(x) \, | f(x)|^p \, dx \leqslant ch^n \sum_k \rho^p(x^k) \, | f(x^k)|^p$$
$$+ ch^p \int_{R_n} |(\sup_{|x-z| \leqslant c'} \rho(z) \, | \nabla f(z)|)(x)|^p \, dx, \tag{31}$$

where c and c' are independent of h. Let $0 < r < p$. Using (18) it follows that the second term on the right-hand side can be estimated by

$$ch^p \int_{R_n} | M \, | \rho f |^r \, (x)|^{p/r} \, dx. \tag{32}$$

Again using Hardy's maximal inequality, the last term can be estimated from above by $ch^p \, \| \rho f \|_{L_p}^p$. If h is sufficiently small, $h \leqslant h_0 \leqslant 1$, then one obtains the right-hand side of (29). A small modification shows that the assertion is also true for $p = \infty$ (instead of (18) one has to use (26) for $p = q = \infty$ and $\partial/\partial x_j$). The left hand side of (29) is proved in the same way by changing the roles of x and x^k in (30).

Step 2. Let $0 < h \leqslant h_0$ and $\mu_1, \mu_2 \in M_h$. To prove (28) for $\kappa_1 = \kappa_2 = \rho$, let f be a function satisfying (13). Since c_1 and c_2 in (29) are independent of the choice of $x^k \in Q_k{}^h$, it follows for $p < \infty$ that

$$\begin{aligned} \int_{R_n} \rho^p(x) \, | f(x)|^p \, d\mu_1 &\leqslant h^n \sum_k \sup_{z \in Q_k{}^h} \rho^p(z) \, | f(z)|^p \\ &\leqslant ch^n \sum_k \inf_{z \in Q_k{}^h} \rho^p(z) \, | f(z)|^p \\ &\leqslant c' \int_{R_n} \rho^p(x) \, | f(x)|^p \, d\mu_2 . \end{aligned} \tag{33}$$

A corresponding estimate holds for $p = \infty$.

Step 3. Let $\mu \in M_h$, where $h \leqslant h_0$, and let $\kappa(x)$ be an admissible function with respect to ρ and μ. We shall show

$$\| \rho f \|_{L_{p,\mu}} \leqslant c \| \kappa f \|_{L_{p,\mu}}. \tag{34}$$

If G has the meaning of Definition 2.3(a) (with respect to μ), then we construct a new measure $\nu \in M_h$ by

$$\nu(\Omega) = \sum_k (\mu(\Omega \cap G \cap Q_k^h)/\mu(G \cap Q_k^h)), \tag{35}$$

where Ω is an arbitrary Borel-measurable set in R_n. If δ has the meaning of Definition 2.3(a) (with respect to μ), then it follows from (33) (with $\mu_1 = \mu$ and $\mu_2 = \nu$) that for $p < \infty$

$$\int_{R_n} \rho^p(x) \, | f(x)|^p \, d\mu \leqslant c/\delta \int_{R_n} \rho^p(x) \, \chi_G{}^p(x) \, | f(x)|^p \, d\mu, \tag{36}$$

where $\chi_G(x)$ is the characteristic function of G. Equation (34) follows from (36) and the fact that $\rho(x) \, \chi_G(x) \leqslant c\kappa(x)$. A corresponding assertion holds for $p = \infty$. Now, (28) is an easy consequence of (33) and (34).

Remark. Equation (28) and its special case (29) show that the inequalities proved here have a lattice structure. The question arises how to understand the restriction $h \leqslant h_0$ of the lattice constant. In Section 3.7, it will be proved that (29) cannot be true if h is too large.

3.5. *The Main Inequality*

THEOREM 3.5. *Let $\rho \in K(a, C_\epsilon)$, $b > 0$, $0 < p \leqslant q \leqslant \infty$, and let μ_1, $\mu_2 \in M_h$ where $0 < h \leqslant h_0$ (here h_0 has the same meaning as in Theorem* 3.4). *Furthermore, let $\kappa_j(x)$ be an admissible function with respect to ρ and μ_j, $j = 1, 2$ (cf. Definition* 2.3(a)). *If α is a multi-index, then there exists a positive number C such that for all functions f of type* (13)

$$\| \kappa_1 D^\alpha f \|_{L_{q,\mu_1}} \leqslant C \| \kappa_2 f \|_{L_{p,\mu_2}}. \tag{37}$$

Proof. If f satisfies (13), then $D^\alpha f$ has the same properties. But now (37) is a consequence of (26) and (28).

EXAMPLE. By (37) it is possible to compare the Lebesgue-measure with the atomic measures as described in the Remark in Section 2.2. Other interesting examples may be obtained on the basis of the examples in Section 2.3. One has

$$\left\| \prod_{j=1}^n | x_j |^{\beta_j} D^\alpha f \right\|_{L_q} \leqslant \left\| \prod_{j=1}^n (1 + | x_j |)^{\beta_j} D^\alpha f \right\|_{L_q} \leqslant C \left\| \prod_{j=1}^n | x_j |^{\beta_j} f \right\|_{L_p},$$

where $0 < p \leqslant q \leqslant \infty$ and $\beta_j \geqslant 0$; and

$$\| \, | x |^{\beta} D^{\alpha} f \|_{L_q} \leqslant \|(1 + | x |^{\beta}) D^{\alpha} f \|_{L_q} \leqslant C \| \, | x |^{\beta} f \|_{L_p},$$

where $0 < p \leqslant q \leqslant \infty$ and $\beta \geqslant 0$. Here $\| \cdot \|_{L_p}$ are the usual spaces with respect to the Lebesgue measure.

3.6. *Extension of Inequality* (37)

For the later applications we extend (37).

THEOREM 3.6. *If ρ, p, q, μ_1, μ_2, κ_1, κ_2, and α have the same meaning as in Theorem* 3.5, *and if Ω is a bounded subset of R_n, then there exists a positive number C such that* (37) *holds for all $f \in L_p^{\Omega}(\kappa_2, \mu_2)$ provided that the lattice constant h is sufficiently small* (*that means $h \leqslant h_0$ where h_0 depends on Ω*).

Proof. Let $f \in L_p^{\Omega}(\kappa_2, \mu_2)$. By Lemma 5.4 and Section 5.6 it follows that f can be approximated in $(S_a)'$ by $f_\delta \in S_a$ such that $\operatorname{supp} Ff_\delta \subset \{y \mid |y| \leqslant b\}$, where b is sufficiently large. Apply (37) to f_δ. If $\alpha = (0,..., 0)$, then one obtains the desired inequality for $\delta \downarrow 0$ (here one uses the explicit form of f_δ as described in (60), after appropriate changes of notations). The general case follows by mathematical induction with respect to $|\alpha|$.

Remark. We extend another inequality which will be useful for later applications. For all functions f, satisfying (13), it follows by (18), (24), and Hardy's maximal inequality that

$$\left\| \sup_z \rho(x - z) \frac{|f(x - z)|}{1 + |z|^{n/r}} \right\|_{L_p} \leqslant C \| \rho f \|_{L_p},$$

where $0 < r < p < \infty$. Using the above approximation argument (and Fatou's lemma), it follows that the last inequality holds true for $f \in L_p^{\Omega}(\rho, \mu_L)$, μ_L indicating the Lebesgue measure.

3.7. *Noninequalities*

The above inequalities have two characteristic features: (i) the lattice structure, which means the arbitrariness of the measures $\mu \in M_h$ in the inequalities (28) and (37), and (ii) the growth conditions for $\rho \in K(a, C_\epsilon)$, expressed by (4). In part (a) of the theorem below and in the remark below we clarify the lattice structure. Parts (b) and (c) of the theorem below show that one cannot weaken essentially the growth condition (4) in the theory developed above.

THEOREM 3.7. (a) *Let* $\rho \in K(a, C_\epsilon)$, $b > 0$, *and* $0 < p \leqslant \infty$. *If* $h > \pi n^{1/2}/b$, *then there does not exist a positive number* C *such that for all choices* $x^k \in Q_k{}^h$ *and all* f, *satisfying* (13), *one has*

$$\| \rho f \|_{L_p} \leqslant C \| \rho(x^k) f(x^k) \|_{l_p} . \tag{38}$$

(b) *If* $f \in (S_a)'$ *such that* supp Ff *is compact and*

$$\| e^{\epsilon|x|} f(x) \|_{L_\infty} \leqslant \infty$$

for an appropriate positive number ϵ, *then* $f(x) \equiv 0$.

(c) *Let* $\epsilon > 0$, $0 < p < \infty$, *and* $0 < a < 1$. *Then there does not exist a positive number* C *such that for all* $f \in S$ (*Schwartz space*) *with* supp $Ff \subset \{y \mid |y| \leqslant \epsilon\}$ *one has*

$$\| e^{-|x|^{1/a}} \Delta f \|_{L_p} \leqslant C \| e^{-|x|^{1/a}} f \|_{L_p} . \tag{39}$$

Proof. (a) Let $0 < b' < b/n^{1/2}$ such that $h > \pi/b'$, and use the known formula

$$F\left(\prod_{j=1}^{n} ((\sin b' x_j)/x_j)) \right) = c\chi', \tag{40}$$

where χ' is the characteristic function of $Q' = \{\xi \mid \xi = (\xi_1, \ldots, \xi_n), |\xi_j| \leqslant b'\}$. If $\varphi \in S_a$, then

$$f(x) = \varphi(x) \prod_{j=1}^{n} ((\sin b' x_j)/x_j)) \in S_a .$$

The remarks in Section 5.3 show that φ can be chosen such that $\varphi(x) \not\equiv 0$ and

$$\operatorname{supp} Ff \subset \{\eta \mid |\eta| \leqslant \delta\}. \tag{41}$$

If $\delta > 0$ is sufficiently small, then it follows that

$$\operatorname{supp} Ff = \operatorname{supp}\left(F\varphi * F \prod_{j=1}^{n} \frac{\sin b' x_j}{x_j} \right) \subset Q' + \{y \mid |y| \leqslant \delta\} \subset \{y \mid |y| \leqslant b\}.$$

Consequently, f satisfies (13). Since $h > \pi/b'$, one may choose for $x^k \in Q_k{}^h$ a subset of the roots $(\pi/b')(l_1, \ldots, l_n)$ of f, where $l_j = \pm 1, \pm 2, \pm 3, \ldots$. Hence, for such a choice of x^k the right-hand side of (38) vanishes. This proves (a).

(b) If f has the described properties, then

$$(Ff)(\zeta) = c \int_{R_n} e^{-i\langle \zeta, x\rangle} f(x)\, dx$$

can be extended to complex ζ, for instance to real, $\zeta_1, \ldots, \zeta_{n-1}$ and complex ζ_n with $|\operatorname{Im} \zeta_n| < \epsilon$. Hence, Ff is an analytic function in the strip $|\operatorname{Im} \zeta_n| < \epsilon$. Since Ff has compact support in R_n, it follows that $Ff \equiv 0$, and thus $f \equiv 0$.

(c) Assume that there exists a constant C such that (39) is true for all $f \in S$ with $\operatorname{supp} Ff \subset \{y \mid |y| \leqslant \epsilon\}$. Let $f \in S'$, $\operatorname{supp} Ff \subset \{y \mid |y| \leqslant \epsilon/2\}$. Using the approximation argument of Lemma 5.4 with respect to S and S' it follows that (39) holds also for f (with the same constant C). In particular, (39) is true for arbitrary polynomials. Let $f(x) = \sum_{j=1}^{n} x_j^m$ where m is a positive even number. Let $r = |x|$. If $p < \infty$, it follows that

$$m^{2p} \int_0^\infty \exp(-pr^{1/a})\, r^{pm-2p+n-1}\, dr \leqslant c \int_0^\infty \exp(-pr^{1/a})\, r^{pm+n-1}\, dr,$$

where c is independent of m. Using the transformation $pr^{1/a} = t$, it follows that

$$m^{2p}\Gamma(apm - 2ap + an) \leqslant c'\Gamma(apm + an), \tag{42}$$

where Γ is Euler's Γ-function. Here c' is independent of m. As a consequence of Stirling's formula one obtains

$$\Gamma(apm + an) \leqslant cm^{2ap}\Gamma(apm - 2ap + an).$$

Because $a < 1$, this is a contradiction to (42).

Remark. Part (a) shows that one cannot expect inequalities of type (28) and (37) if the lattice constant h is too large. But we add here a formula which gives a better understanding of the lattice character of the above inequalities if h is sufficiently small. Let (for simplicity) $f \in S$ and

$$\operatorname{supp} Ff \subset Q_b = \{\xi \mid |\xi_j| \leqslant b\},$$

Then we have the Fourier expansion ($kx = \sum_{j=1}^{n} k_j x_j$)

$$(Ff)(x) = \chi(x) \sum_k a_k \exp(-i\pi kx/b),$$

$\chi(x)$ being the characteristic function of Q_b. Here

$$a_k = \frac{1}{b^n} \int_{Q_b} (Ff)(x) \exp(i\pi kx/b)\, dx = \frac{c}{b^n} [F^{-1}(Ff)]\left(\frac{\pi}{b} k\right) = \frac{c}{b^n} f\left(\frac{\pi}{b} k\right)$$

Hence,

$$\begin{aligned} f(x) &= \frac{c'}{b^n} \sum_k f\left(\frac{\pi}{b} k\right) F^{-1}(\chi \exp(-i\pi k\xi/b))(x) \\ &= \frac{c'}{b^n} \sum_k f\left(\frac{\pi}{b} k\right) (F^{-1}\chi)\left(x - \frac{\pi}{b} k\right). \end{aligned} \tag{43}$$

This shows that the values of f in the lattice points $\pi k/b$ determine $f(x)$ completely.

4. The Spaces $L_p^\Omega(\kappa, \mu)$

The spaces $L_p^\Omega(\kappa, \mu)$ were defined in (12). A first result for these spaces was obtained in Theorem 3.6. The main aim of this section is to show that all these spaces are quasi-Banach spaces.

4.1. *Quasi-Banach Spaces*

Theorem 4.1. (a) *All the spaces $L_p^\Omega(\kappa, \mu)$ as given by Definition 2.3(b) are quasi-Banach spaces (for $p \geqslant 1$ Banach spaces), provided that h is sufficiently small ($h \leqslant h_0$, where h_0 has the meaning of Theorem 3.6).*

(b) *Let $\rho \in K(a, C_\epsilon)$ and $\mu_1, \mu_2 \in M_h$, where $h \leqslant h_0$. Then*

$$L_p^\Omega(\kappa_1, \mu_1) = L_p^\Omega(\kappa_2, \mu_2) \tag{44}$$

provided that $0 < p \leqslant \infty$, Ω is a bounded subset of R_n, and $\kappa_j(x)$ is an admissible function with respect to ρ and μ_j ($j = 1, 2$).

Proof. (a) Let $f \in L_p^\Omega(\kappa, \mu)$ and $\| \kappa f \|_{L_{p,\mu}} = 0$. Then $\| \rho f \|_{L_\infty} = 0$, by Theorem 3.6, and consequently $f(x) \equiv 0$ (here ρ is the corresponding function from Definition 2.3). This proves that $L_p^\Omega(\kappa, \mu)$ is a quasi-normed space. To prove completeness, let $\{f_j\}_{j=1}^\infty$ be a fundamental sequence in $L_p^\Omega(\kappa, \mu)$. Again by Theorem 3.6 it follows that $\{\rho f_j\}_{j=1}^\infty$ is a fundamental sequence in L_∞. By (4), $\{\exp(-\epsilon \mid x \mid^{1/a}) f_j\}_{j=1}^\infty$ is a fundamental sequence in L_∞ for each positive ϵ. Using the argumentation in Lemma 5.1, it follows that $\{f_j\}_{j=1}^\infty$ is a fundamental sequence in $(S_a)'$, and hence $\{Ff_j\}_{j=1}^\infty$ is a fundamental sequence in $(S^a)'$. Now it is not hard to see that there exists a function f such that

$$\| \kappa f - \kappa f_j \|_{L_{p,\mu}} \to 0 \qquad \text{and} \qquad Ff_j \xrightarrow[(S^a)']{} Ff \qquad \text{for} \quad j \to \infty.$$

In particular, $\operatorname{supp} Ff \subset \bar{\Omega}$. Hence, $f \in L_p^\Omega(\kappa, \mu)$, which proves the completeness.

(b) Equation (44) is an immediate consequence of Theorem 3.6.

4.2. *Density Property*

A bounded domain Ω in R_n has the segment property if there exist open balls B_j and vectors $y^j \in R_n$, $j = 1,\ldots, N$, such that

$$\bigcup_{j=1}^N B_j \supset \bar{\Omega} \qquad \text{and} \qquad (\bar{\Omega} \cap B_j) + ty^j \subset \Omega$$

for $0 < t \leqslant 1$ and $j = 1,\ldots, N$.

THEOREM 4.2. *Let Ω be a bounded domain with segment property. Let $L_p{}^{\Omega}(\kappa, \mu)$ be the space described in Definition* 2.3(b). *If h is sufficiently small* ($h \leqslant h_0$, *where h_0 has the meaning of Theorem* 3.6) *and if $p < \infty$, then*

$$\{f \mid f \in S_a\,,\ \operatorname{supp} Ff \subset \Omega\}$$

is dense in $L_p{}^{\Omega}(\kappa, \mu)$.

Proof. Let $\{\varphi_j(x)\}_{j=1}^N \subset S_a$ be such that $\{F\varphi_j(x)\}_{j=1}^N$ is a partition of unity with respect to the balls B_j,

$$\sum_{j=1}^{N} F\varphi_j(x) = 1 \qquad \text{for} \quad x \in \bar{\Omega}, \quad F\varphi_j(x) \in S^a \tag{45}$$

(the remarks in Section 5.3 ensure the existence of such a partition of unity). By (49) and (62)

$$(\Phi_j f)(x) = c \int_{R_n} \varphi_j(x-y) f(y)\, dy, \qquad j = 1,\ldots, N, \tag{46}$$

is meaningful for $f \in L_p{}^{\Omega}(\kappa, \mu)$. By a suitable choice of c one has $\sum_{j=1}^N (\Phi_j f)(x) = f(x)$, which is a consequence of (45) and an application of the Fourier transform to (46). We want to show that Φ_j is a bounded operator in $L_p{}^{\Omega}(\kappa, \mu)$. By Theorem 4.1 we may assume that μ is the Lebesgue measure and that $\kappa = \rho$. Repeating the arguments at the beginning of the proof to Lemma 3.2 it follows that

$$\begin{aligned} \rho(x)\, |(\Phi_j f)(x)| &\leqslant \sup_z \rho(x-z) \frac{|(\Phi_j f)(x-z)|}{1 + |\, z\,|^{n/r}} \\ &\leqslant c \sup_z \rho(x-z) \frac{|\, f(x-z)|}{1 + |\, z\,|^{n/r}}\,. \end{aligned}$$

The remark in Section 3.6 shows that Φ_j is a bounded operator in $L_p{}^{\Omega}(\rho, \mu)$. Hence it will be sufficient to approximate $\Phi_j f$. Let, without loss of generality, $\Phi_j f = f$. If y^j has the above meaning, then $f^t(x) = f(x)\, e^{it\langle y^j, x\rangle}$ belongs to $L_p{}^{\Omega}(\rho, \mu)$ provided that $0 < t \leqslant 1$, and one has $(Ff^t)(\xi) = (Ff)(\xi - ity^j)$. In particular, $\operatorname{supp} Ff^t \subset \Omega$, the support of Ff^t having a positive distance to the boundary of Ω. Now $f^t \to f$ in $L_p{}^{\Omega}(\rho, \mu)$ for $t \downarrow 0$. But f^t can be approximated by the method described in the proof of Theorem 3.6, completing the proof.

5. APPENDIX: ULTRA-DISTRIBUTIONS AND ENTIRE ANALYTIC FUNCTIONS OF EXPONENTIAL TYPE

Here we present some facts of the theory of ultra-distributions used in this paper. The main result is an extension of the well-known Paley–Wiener–Schwartz theorem.

5.1. *The Spaces S_a and $(S_a)'$*

Essentially we shall use the notations by Gelfand and Schilow [2, Chap. IV]. If $1 < a < \infty$, then S_a denotes the set of all complex-valued infinitely differentiable functions $\varphi(x)$ defined on the n-dimensional real Euclidean space R_n such that for all multi-indices α and all integers $k = 0, 1, 2,...,$

$$\sup_{x \in R_n} |x|^k |D^\alpha \varphi(x)| \leqslant C^{(1)}_{|\alpha|} A^k k^{ak}. \tag{47}$$

Here A and $C^1_{|\alpha|}$ are appropriate positive numbers, depending on $\varphi(x)$. If A is fixed, then $S_{a,A}$ denotes the locally convex space whose topology is generated by the norms

$$\| \varphi \|_{l,A+\delta} = \sup_{\substack{x \in R_n \\ k=0,1,2,...}} \frac{|x|^k \sum_{|\alpha| \leqslant l} |D^\alpha \varphi(x)|}{(A+\delta)^k k^{ak}}. \tag{48}$$

Here $\delta > 0$ and $l = 0, 1, 2,...$. Obviously $S_a = \bigcup_{A>0} S_{a,A}$. Now, S_a becomes a locally convex space if it is considered as the inductive limit of $S_{a,A}$ (a short description of inductive limits of locally convex spaces and the facts needed here may be found in [5, III, 7]). Let $(S_a)'$ and $(S_{a,A})'$ be the topological duals of S_a and $S_{a,A}$, respectively. The definition of the inductive limit yields that f belongs to $(S_a)'$ if and only if the restriction of f to $S_{a,A}$ belongs to $(S_{a,A})'$ for all $A > 0$. The following characterization of S_a is of interest: A complex-valued infinitely differentiable function $\varphi(x)$ belongs to S_a if and only if there exist positive numbers B and $C^{(2)}_l$ such that for all multi-indices α

$$|D^\alpha \varphi(x)| \leqslant C^{(2)}_{|\alpha|} \exp(-B |x|^{1/a}). \tag{49}$$

The norms (48) may be replaced by the equivalent norms

$$\| \varphi \|'_{l,B-\delta} = \sup_{x \in R_n} \exp((B-\delta) |x|^{1/a}) \sum_{|\alpha| \leqslant l} |D^\alpha \varphi(x)|. \tag{50}$$

Here $\delta > 0$ and $l = 0, 1, 2,...$. A proof may be found in [2], where also the dependence between A in (48) and B in (50) is given (in [2, IV], the one-dimensional case is treated, but the considerations can be extended to the n-dimensional case; see also [2, IV, Section 9]). The following simple conclusion of the last remarks is useful.

LEMMA 5.1. *Let $1 < a < \infty$ and $f(x)$ be a Borel-measurable function in R_n such that for each positive number ϵ there exists a positive number $c(\epsilon)$ with*

$$|f(x)| \leqslant c(\epsilon) \exp(\epsilon |x|^{1/a}).$$

Then f belongs to $(S_a)'$ (with the usual interpretation).

Proof. The proof follows from (50) and

$$\left| \int_{R_n} f(x)\, \varphi(x)\, dx \right| \leqslant c \sup_{x\in R_n} (1 + | x |^{n+1}) \exp(\epsilon \, | x |^{1/a}) \, | \varphi(x)| \leqslant c' \| \varphi \|'_{0,2\epsilon} .$$

5.2. *The Spaces S^a and $(S^a)'$*

If $1 < a < \infty$, then S^a denotes the set of all complex-valued infinitely differentiable functions $\varphi(x)$ defined in R_n such that for all multi-indices α and all integers $k = 0, 1, 2,...,$

$$\sup_{x\in R_n} | x |^k \, | D^\alpha \varphi(x)| \leqslant C_k^{(3)} A^{|\alpha|} \, | \alpha |^{a|\alpha|}. \tag{51}$$

Here A and $C_k^{(3)}$ are appropriate positive numbers depending on $\varphi(x)$. We introduce a topology in S^a in completely the same manner as in S_a: The space S^a is the inductive limit of the locally convex spaces $S^{a,A}$ where the topology in these spaces is generated by the norms

$$\| \varphi \|''_{k,A+\delta} = \sup_{\substack{x\in R_n \\ l=0,1,2,\ldots}} \frac{| x |^k \sum_{|\alpha|\leqslant l} | D^\alpha \varphi(x)|}{(A + \delta)^l \, l^{al}} . \tag{52}$$

Here $\delta > 0$ and $k = 0, 1, 2,...$. S^a are spaces of Gevrey type, (see [4, Chap. 7, 1.2]). Let $(S^a)'$ be the topological dual of S^a.

5.3. *Properties of S_a, $(S_a)'$, S^a, and $(S^a)'$*

The definition of S_a yields

$$D(R_n) \subset S_a(R_n) \subset S(R_n), \tag{53}$$

where $D = D(R_n)$ is the space of all complex-valued infinitely differentiable functions in R_n with compact support, and $S = S(R_n)$ is the usual Schwartz space of rapidly decreasing functions. If D and S are equipped with the usual topologies (in particular $D(R_n)$ is the inductive limit of $D(K_N) = C_0^\infty(K_N)$ where $K_N = \{x \mid | x | \leqslant N\}$, $N \to \infty$), then (53) also holds in the topological sense. It is not hard to see that the embeddings in (53) are dense embeddings. Consequently, by the usual interpretation,

$$S' \subset (S_a)' \subset D'. \tag{54}$$

Now we collect some properties of the spaces S^a (see [8, 9]; a short description is also given in [4, Chap. 7, 1.2]). If ω_1 and ω_2 are two bounded (open) domains such that $\bar{\omega}_2 \subset \omega_1$, then there exists a function $\varphi \in S^a$ such that

$$\operatorname{supp} \varphi \subset \omega_1 \qquad \text{and} \qquad \varphi(x) = 1 \quad \text{for } x \in \bar{\omega}_2 . \tag{55}$$

We may assume $\varphi(x) \geqslant 0$ for $x \in R_n$. Further, if $\bar{\omega}_2$ is covered by a finite number of open balls, then there exists a corresponding partition of unity by functions belonging to S^a. One may compare the space S^a with the space $D_a = D_{M_l}(R_n)$ with $M_l = l^{al}$, considered in [4, Chap. 7, p. 2] (see also [8, 9]). There holds (in the sense of dense topological embedding)

$$D_a \subset S^a.$$

Consequently (by the usual interpretation)

$$(S^a)' \subset D_a',$$

where D_a' are ultra-distributions of Gevrey type. A constructive description of the elements of D_a' is given in [9] and [4, p. 7]. The above remarks on particular functions in S^a show that it is meaningful to define the support of $f \in (S^a)'$ in the usual way.

5.4. *The Fourier Transform*

F denotes the usual Fourier transform in the Schwartz space $S = S(R_n)$, its inverse is F^{-1}. Both the spaces S_a and S^a are subspaces of S. Consequently, the restriction of F and F^{-1} to these spaces is meaningful. One has the following fundamental fact:

$$FS_a = S^a, \qquad FS^a = S_a. \tag{56}$$

This means that F is a one-to-one map from S_a onto S^a (resp. from S^a onto S_a), continuously in both directions. The same holds for F^{-1}. A proof may be found in [2, IV, Section 6]. By the usual procedure

$$(Ff)(\varphi) = f(F\varphi),$$

the Fourier transform and its inverse can be carried over onto $(S_a)'$ and $(S^a)'$. Then

$$F(S_a)' = (S^a)', \qquad F(S^a)' = (S_a)'. \tag{57}$$

In particular, (54) shows that the Fourier transform may be extended to more general spaces than S', the space of tempered distributions.

The following approximation procedure is useful. Let $\varphi(x) \in D$,

$$\varphi(x) \geqslant 0, \qquad \int_{R_n} \varphi(x)\,dx = 1, \qquad \operatorname{supp} \varphi \subset \{y \mid |y| \leqslant 1\}. \tag{58}$$

Let $\varphi_h(x) = h^{-n}\varphi(x/h)$, where $h > 0$. Then $(F\varphi_h)(\xi) = (F\varphi)(h\xi)$ and $(F\varphi_h)(0) = 1$. Using (50) it follows that for $\psi \in S_a$

$$\psi F\varphi_h \xrightarrow[S_a]{} \psi \qquad \text{for} \quad h \downarrow 0. \tag{59}$$

One can extend the last assertion to $(S_a)'$. For our purpose it will be sufficient to equip $(S_a)'$ with the weak topology.

LEMMA 5.4. *Let $f \in (S_a)'$. Then*

$$(F\varphi_h) f \xrightarrow[(S_a)']{} f \qquad \text{for} \quad h \downarrow 0, \tag{60}$$

$$\operatorname{supp} F((F\varphi_h) f) \subset \operatorname{supp} Ff + \{x \mid |x| \leqslant h\}. \tag{61}$$

Proof. Explaining (60) in the usual way:

$$[(F\varphi_h) f](\psi) = f(\psi F\varphi_h), \qquad \psi \in S_a,$$

(60) is a consequence of (59). If $\chi \in S^a$, then

$$\begin{aligned} F((F\varphi_h) f)(\chi) &= f(F\varphi_h \cdot F\chi) \\ &= Ff(F^{-1}(F\varphi_h F\chi)) = Ff(\varphi_h * \chi). \end{aligned}$$

Thus (61) follows from $\operatorname{supp}(\varphi_h * \chi) \subset \operatorname{supp} \chi + \operatorname{supp} \varphi_h$.

5.5. *A Theorem of Paley–Wiener–Schwartz Type*

The classical Paley–Wiener–Schwartz theorem gives a characterization of tempered distributions f with the additional property that the support of the Fourier transform Ff is compact (cf. [3, Theorem 1.7.7]). We extend this theorem to distributions belonging to $(S_a)'$. For brevity we introduce the following notation: Let $1 < a < \infty$ and $b > 0$. A complex-valued function $f(x)$ defined in R_n is said to be of type (a, b) if there exists an extension $f(z)$ of $f(x)$ to the n-dimensional complex space C_n with the following properties:

(i) $f(z)$ is an entire analytic function in C_n,

(ii) for each positive ϵ there exists a positive number c_ϵ such that

$$|f(z)| \leqslant c_\epsilon \exp(\epsilon |z|^{1/a}) \exp((b + \epsilon) |\operatorname{Im} z|). \tag{62}$$

Here $|\operatorname{Im} z| = (\sum_{j=1}^n |\operatorname{Im} z_j|^2)^{1/2}$ for $z = (z_1, ..., z_n) \in C_n$. In particular, $f(z)$ is of exponential type.

THEOREM 5.5. *The following two assertions are equivalent*:

(a) $f \in (S_a)'$ *and* $\operatorname{supp} Ff \subset \{y \mid |y| \leqslant b\}$,

(b) f is of type (a, b) (via the usual interpretation of functions as distributions).

Proof. Step 1. Let $f \in (S_a)'$ and $\operatorname{supp} Ff \subset \{y \mid |y| \leqslant b\}$. Then $g = Ff \in (S^a)'$. Let $\epsilon > 0$. We choose a function $\chi(x) \in S^a$ with $\operatorname{supp} \chi \subset \{y \mid |y| \leqslant b + \epsilon\}$ such that $\chi(x) = 1$ for $|x| \leqslant b + \epsilon/2$ (the remarks in Section 5.3 ensure the existence of such a function). Then $e^{i\langle x, \xi\rangle}\chi(x) \in S^a$

where $\xi \in C_n$. Here $\langle \cdot, \cdot \rangle$ denotes the scalar product. First we consider the function

$$h(\xi) = g(\chi(\cdot)\, e^{i\langle \cdot, \xi \rangle})$$

on C_n. Using the norms (52) with an appropriate chocie of A it follows that $h(\xi)$ has first derivatives with respect to the complex variables $\xi_1, ..., \xi_n$. Hence $h(\xi)$ is an entire analytic function in C_n. Furthermore, for all $A > 0$ one has again by (52) that

$$| h(\xi) | \leqslant c \sup_{\substack{x \in R_n \\ l=0,1,2,...}} \frac{\sum_{|\alpha| \leqslant l} | D_x^{\alpha}(\chi(x)\, e^{i\langle x, \xi \rangle}) |}{A^l l^{al}}$$

(D_x^{α} indicates the differentiation with respect to x). Let

$$\sup_{x \in R_n} | D^{\beta}\chi(x) | \leqslant c B^m m^{am} \qquad \text{for} \quad | \beta | \leqslant m.$$

(the dependence of $\chi(x)$ upon k in (51) is unimportant because $\chi(x)$ has a compact support). Using the last estimates it follows that

$$\begin{aligned} | h(\xi) | &\leqslant c \sup_{\substack{x \in R_n \\ m \leqslant l}} 3^l \, | \xi |^{l-m} \, e^{|x| |\mathrm{Im}\, \xi|} \sum_{|\beta| \leqslant m} | D^{\beta}\chi(x) | \, A^{-l} l^{-al} \\ &\leqslant c' e^{(b+\epsilon)|\mathrm{Im}\,\xi|} \sup_{m \leqslant l} \left(\frac{4B}{A}\right)^l \left(\frac{m^m (l-m)^{l-m}}{l^l}\right)^a \left(\frac{| \xi |^{(l-m)/a}}{B^{(l-m)/a}(l-m)!}\right)^a. \end{aligned}$$

Without loss of generality we may assume that B is sufficiently large, namely $B \geqslant 1$ and a $B^{-1/a} \leqslant \epsilon$. Choosing A sufficiently large, namely $A \geqslant 4B$, it follows that

$$| h(\xi) | \leqslant c_{\epsilon} \exp(\epsilon \, | \xi |^{1/a}) \exp((b + \epsilon) \, | \mathrm{Im}\, \xi \, |). \tag{63}$$

Now we prove part (a). Let $\varphi \in S_a$. Then it follows that

$$f(\varphi) = g(F^{-1}\varphi) = cg\left(\chi(x) \int_{R_n} e^{i\langle x, \xi \rangle} \varphi(\xi)\, d\xi\right).$$

Using (52) and (63), one can prove (by approximating the integral by finite sums)

$$f(\varphi) = c \int_{R_n} h(\xi)\, \varphi(\xi)\, d\xi.$$

Therefore, $f = ch(\xi)$ by (63) and Lemma 5.1.

Step 2. It will be useful to sharpen the assertion of the first step for the case $f \in S_a$ and $\operatorname{supp} Ff \subset \{y \mid | y | \leqslant b\}$. One has $Ff \in S^a$. The extension of f to C_n is given by

$$f(\xi) = c \int_{R_n} e^{i\langle x, \xi \rangle} (Ff)(x)\, dx = c \int_{R_n} e^{i\langle x, \mathrm{Re}\,\xi \rangle} e^{-\langle x, \mathrm{Im}\,\xi \rangle} (Ff)(x)\, dx.$$

We have $e^{-\langle x, \operatorname{Im} \xi \rangle}(Ff)(x) \in S^{a,B}$ where B is independent of ξ. The Fourier transform maps $S^{a,B}$ onto $S_{a,B}$ (see [2, IV, Section 6.2]). In particular, (50) and (52) yield

$$\exp(A \mid \operatorname{Re} \xi \mid^{1/a}) \mid f(\xi) \mid \leqslant c(1 + \mid \operatorname{Im} \xi \mid^{l}) \exp(b \mid \operatorname{Im} \xi \mid),$$

where c is independent of $\xi \in C_n$, and A and l are appropriate positive numbers. It follows that

$$\begin{aligned} \mid f(\xi) \mid &\leqslant c_\epsilon' \exp(-A \mid \operatorname{Re} \xi \mid^{1/a}) \exp((b + (\epsilon/2)) \mid \operatorname{Im} \xi \mid) \\ &\leqslant c_\epsilon'' \exp(-A \mid \xi \mid^{1/a}) \exp((b + \epsilon) \mid \operatorname{Im} \xi \mid). \end{aligned} \tag{64}$$

This is the counterpart to (63).

Step 3. Let f be of type (a, b). Lemma 5.1 shows that f belongs to $(S_a)'$. We must prove the assertion concerning the support of Ff. Let φ_h be the function of Lemma 5.4, where we additionally assume $\varphi_h \in S^a$. The second step, where F is replaced by F^{-1}, may be applied to $F\varphi_h$. Then (64) holds, with f replaced by $F\varphi_h$ and b by h. Hence it follows for the entire analytic function $f(z)(F\varphi_h)(z)$ that

$$\begin{aligned} \mid f(z)(F\varphi_h)(z) \mid &\leqslant c \exp(-C \mid z \mid^{1/a}) \exp((b + h + 2\epsilon) \mid \operatorname{Im} z \mid) \\ &\leqslant \frac{c'}{(1 + \mid z \mid)^N} \exp(b + h + 2\epsilon) \mid \operatorname{Im} z \mid), \end{aligned} \tag{65}$$

where N is an arbitrary positive number. But this is the classical situation (see [3, 1.7.7]). Consequently,

$$\operatorname{supp} F(fF\varphi_h) \subset \{y \mid \mid y \mid \leqslant b + h\}.$$

Using Lemma 5.4 it follows that $\operatorname{supp} Ff \subset \{y \mid \mid y \mid \leqslant b + h\}$. Because h is an arbitrary positive number, one obtains $\operatorname{supp} Ff \subset \{y \mid \mid y \mid \leqslant b\}$.

Remark. In connection with the last theorem we refer also to [9, Theorem 21], where a similar result may be found.

5.6. *Approximation Property*

With the aid of the last theorem one can sharpen Lemma 5.4. Let f be of type (a, b). Again let $\varphi_h \in S^a$ be the functions of Lemma 5.4. For an arbitrary multi-index γ, $\psi \to D^\gamma \psi$ is a linear and continuous map from S_a into S_a and from S^a into S^a, and so also from $(S_a)'$ into $(S_a)'$ and from $(S^a)'$ into $(S^a)'$. But then the last theorem can be applied to $D^\gamma f$ and $D^\gamma F\varphi_h$. Using (65) for these modified functions it follows that $(F\varphi_h)f \in S_a$. Consequently, (60) is an approximation by entire analytic functions belonging to S_a.

Note added in proof. In connection with ultra-distributions (and in particular with Theorem 5.5) we also refer to: G. Björck, Linear partial differential operators and generalized distributions, *Ark. Mat.* **6** (1966), 351–407.

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